Auditorium Exercise Sheet 7

Differential Equations I for Students of Engineering Sciences

Eleonora Ficola

Department of Mathematics of Hamburg University

Winter Semester 2023/2024

15.01.2024

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Laplace transform

Given a function $f : [0, \infty) \to \mathbb{R}$, consider the mapping $F_f : D \subseteq \mathbb{C} \to \mathbb{C}$ such that

$$F_f[s] := \mathcal{L}[f(t)] := \int_0^\infty e^{-st} f(t)\; dt.$$

We call F_f the **Laplace transformation** of f, while for all s such that the integral converges the function $F_f[s] = \mathcal{L}[f(t)]$ is the **Laplace transform** of f at point $t \geq 0$.

We denote the correspondence between f and its Laplace transform F_f (from now on simply F) by the Doetsch symbol:

$$f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s) \qquad F(s) \bullet\!\!-\!\!\!-\!\!\circ f(t)$$

Some properties of Laplace transform

- Linearity: $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$, for all $a, b \in \mathbb{C}$
- Transformation of derivatives: for $k \in \mathbb{N}_0$,

  $\mathcal{L}[f^{(k)}(t)] = s^k\mathcal{L}[f(t)] - s^{k-1}f(0) - s^{k-2}f'(0) - \ldots - sf^{k-2}(0) - f^{k-1}(0)$,

  in particular: $\boldsymbol{\mathcal{L}[f'(t)] = s \cdot \mathcal{L}[f(t)] - f(0)}$
  and $\boldsymbol{\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - s \cdot f(0) - f'(0)}$
- Multiplication rule: $\mathcal{L}\left[t^k f(t)\right] = (-1)^k F^{(k)}(s)$, for $k \in \mathbb{N}_0$
- Damping-shifting property: $\mathcal{L}[e^{at} \cdot f(t)] = F(s-a)$, for $a \in \mathbb{C}$
- Scaling property: $\mathcal{L}\left[f(\alpha t)\right] = \frac{F(s/\alpha)}{\alpha}$, for $\alpha > 0$

Laplace transform of the Heaviside function

For any real value a, the **Heaviside function** h_a is the unit step function such that

$$h_a(t) := \begin{cases} 1; & \text{for } t \geq a; \\ 0; & \text{for } t < a. \end{cases}$$

Observe that $h_a(t) = h_0(t-a)$. For $a \geq 0$, it is

$$\mathcal{L}[h_a(t)] = \int_0^\infty e^{-st} h_a(t)\ dt = \int_a^\infty e^{-st}\ dt = \frac{e^{-as}}{s} - \lim_{t\to\infty}\left(\frac{e^{-ts}}{s}\right) = \frac{e^{-as}}{s}.$$

A generalization yields the **shifting property**:

$$\mathcal{L}[h_a(t)f(t-a)] = e^{-sa}\mathcal{L}[f(t)]$$

Table of fundamental Laplace transforms

$f(t) \circ\!\!-\!\!\bullet\, F(s) \qquad F(s) \bullet\!\!-\!\!\circ\, f(t)$

$f(t),\ t \geq 0$	$F(s)$	$s \in \mathbb{C}:\ \Re(s) > \gamma_0$
1	$\frac{1}{s}$	$\gamma_0 = 0$
t^n	$\frac{n!}{s^{n+1}}$	$\gamma_0 = 0$
$h_a(t)$, for $a \in \mathbb{R}_0^+$	$\frac{e^{-as}}{s}$	$\gamma_0 = 0$
e^{at}, for $a \in \mathbb{C}$	$\frac{1}{s-a}$	$\gamma_0 = \Re(a)$
$e^{at}\sin(\omega t)$, for $\omega \in \mathbb{R}$	$\frac{\omega}{(s-a)^2+\omega^2}$	$\gamma_0 = \Re(a)$
$e^{at}\cos(\omega t)$, for $\omega \in \mathbb{R}$	$\frac{s-a}{(s-a)^2+\omega^2}$	$\gamma_0 = \Re(a)$

Existence and uniqueness of Laplace transform

A function $f : [0, \infty) \to \mathbb{R}$ is **exponential of order $\gamma_0 \in \mathbb{R}$** if there is some $C > 0$ such that $|f(t)| \leq Ce^{\gamma_0 t}$, for all $t \geq 0$.

Theorem (Existence of Laplace transform)

If $f : [0, \infty) \to \mathbb{R}$ is exponential of order γ_0 and piece-wise continuous, then its Laplace transform $F = F(s)$ exists for all $s \in \mathbb{C}$ such that $\Re(s) > \gamma_0$.

Theorem (Uniqueness of Laplace transform)

Let $f, g : [0, \infty) \to \mathbb{R}$ exponential of order γ_0 and piece-wise continuous. If $F(s) = G(s)$ for all $s \in \mathbb{C}$ such that $\Re(s) > \gamma_0$, then $f(t) = g(t)$ for all t continuity points of f, g.

From the last theorem we deduce that (under exponential growth and piece-wise continuity) the Laplace transform is invertible!

Laplace transform to solve linear IVPs with constant coefficients

Consider a linear ODE of order $m \in \mathbb{N}$ with constant coefficients and m initial conditions at point $t_0 = 0$, that is

$$\begin{cases} a_m y^{(m)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = b(t) \\ y(0) = z_0,\ y'(0) = z_1,\ \ldots,\ y^{(m-1)}(0) = z_{m-1}, \end{cases} \tag{1}$$

for $a_i, z_i \in \mathbb{R}$ given.

- Applying the Laplace transformation to (1), we obtain a linear *algebraic* equation in $Y(s) := \mathcal{L}[y(t)]$
- Solve this latter for $Y(s)$, then invert the Laplace transform to obtain the solution $y(t)$ of (1)

Example 1 (1)

Determine the solution of the following second order IVP with the help of Laplace transform.

$$\begin{cases} y''(t) - 4y'(t) + 3y(t) = 1; \;\; t > 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \tag{2}$$

From the correspondence $y(t) \circ\!\!-\!\!\bullet\, Y(s)$ for $s \in \mathbb{R}$, by linearity and applying the appropriate formulas we rewrite (2) as

$$\mathcal{L}\left[y''(t)\right] - 4\mathcal{L}\left[y'(t)\right] + 3\mathcal{L}\left[y(t)\right] = \mathcal{L}\left[1\right]$$

$$[s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)}] - 4[sY(s) - \cancel{y(0)}] + 3Y(s) = \frac{1}{s}$$

$$(s^2 - 4s + 3)Y(s) = \frac{1}{s}$$

thus

$$Y(s) = \frac{1}{s(s^2 - 4s + 3)} = \frac{1}{s(s-1)(s-3)}.$$

Example 1 (2)

Apply partial fraction decomposition:

$$Y(s) = \frac{1}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3} = \cdots = \frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)},$$

and observe that

$$\frac{1}{s} \bullet\!\!-\!\!\circ\, 1, \quad \frac{1}{s-1} \bullet\!\!-\!\!\circ\, e^{t}, \quad \frac{1}{s-3} \bullet\!\!-\!\!\circ\, e^{3t} \text{ for all real } s > \max\{0,1,3\} = 3,$$

hence

$$Y(s) = \frac{1}{3}\mathcal{L}[1] - \frac{1}{2}\mathcal{L}[e^{t}] + \frac{1}{6}\mathcal{L}[e^{3t}] = \mathcal{L}\left[\frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6}\right] \underset{\exists!}{\Longrightarrow}$$

$\Longrightarrow$ $y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6}$ is the sol. of the IVP (2).

Equilibrium of autonomous systems

Consider an autonomous system of n ODEs

$$\mathbf{y}'(t) = F(\mathbf{y}(t)), \tag{3}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ differentiable on $I \subseteq \mathbb{R}$.

Definition (equilibria of autonomous systems)

A vector $\mathbf{y}_0 \in \mathbb{R}^n$ such that $F(\mathbf{y}_0) = 0$ is called a **stationary point** (or an **equilibrium point**) of the autonomous system $\mathbf{y}'(t) = F(\mathbf{y}(t))$.

If $\mathbf{y}_0$ stationary point of (3), the function $\tilde{\mathbf{y}}(t) := \mathbf{y}_0$, $\forall t \in I$ is such that $\tilde{\mathbf{y}}'(t) = 0 = F(\mathbf{y}_0) = F(\tilde{\mathbf{y}}(t))$, for every $t \in I \implies \mathbf{y}_0$ stationary sol. of (3).

We wonder if a solution of (3) which is "close" to an equilibrium point $\mathbf{y}_0$ at some initial time t_0 remains "close" to $\mathbf{y}_0$ for larger t.

Definition (types of equilibria)

Let $\mathbf{y}_0 \in \mathbb{R}^n$ be a stationary point for the autonomous system $\mathbf{y}'(t) = F(\mathbf{y}(t))$ defined for $t \in I := (\alpha, \infty)$; $t_0 \in I$. We say $\mathbf{y}_0$ is a point of:

- **uniformly* stable equilibrium** if $\forall \varepsilon > 0, \exists\ \delta > 0$ s.t., provided $|\mathbf{y}(t_0) - \mathbf{y}_0| < \delta$, then $|\mathbf{y}(t) - \mathbf{y}_0| < \varepsilon$ for all $t \geq t_0$;
- **asymptotically stable equilibrium** if $\mathbf{y}_0$ is stable and attractive (i.e. $\exists\ \delta > 0$ s.t. if $|\mathbf{y}(t_0) - \mathbf{y}_0| < \delta$, then $\lim_{t\to\infty} |\mathbf{y}(t) - \mathbf{y}_0| = 0$);
- **unstable equilibrium** if it is not stable.

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* For linear, autonomous systems, all points of stable equilibrium are uniformly stable and viceversa.

Specifically, for linear autonomous systems one makes use of the following necessary and sufficient condition.

Equilibrium of autonomous systems

Theorem (Equilibrium of autonomous systems)

Let $\boldsymbol{y}_0 \in \mathbb{R}^n$ be a stationary point for the system $\boldsymbol{y}'(t) = F(\boldsymbol{y}(t))$ and let $JF(\boldsymbol{y}_0)$ be the Jacobian matrix of F evaluated at point $\boldsymbol{y}_0$, with $\{\lambda_i\}_{i=1,\dots,n}$ eigenvalues of $JF(\boldsymbol{y}_0)$. Then:

- *if $\Re(\lambda_i) < 0$ for every $i \in \{1, \dots, n\} \implies \boldsymbol{y}_0$ is a point of **asymptotically stable** equilibrium;*
- *if there is at least one $i \in \{1, \dots, n\}$ s.t. $\Re(\lambda_i) > 0 \implies \boldsymbol{y}_0$ is a point of **unstable** equilibrium.*

Notice that the Theorem does not cover all possible cases...

Example 2 (1)

Consider the (non-linear) autonomous system of differential equations

$$\begin{cases} {y_1}'(t) = \alpha y_1 + y_2(2 + y_1 y_2) \\ {y_2}'(t) = \alpha y_1(y_1^2 + 1) + y_1 + y_2, \end{cases} \tag{4}$$

where $\alpha \neq -2$ is a real parameter and $t \in \mathbb{R}$.

- We may rewrite (4) as $\mathbf{y}'(t) = F(\mathbf{y}(t))$, with $\mathbf{y}(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and $F(\mathbf{y}(t)) := \begin{pmatrix} \alpha y_1 + y_2(2 + y_1 y_2) \\ \alpha y_1(y_1^2 + 1) + y_1 + y_2 \end{pmatrix}$
- The system is homogeneous: $\mathbf{y}_0 = (0,0)^T$ is a stationary point of (4), regardless on the value of α
- Apply the previous Theorem to determine the stability of $(0,0)^T$

Example 2 (2)

$$F(\mathbf{y}) = \begin{pmatrix} \alpha y_1 + y_2(2 + y_1 y_2) \\ \alpha y_1(y_1^2 + 1) + y_1 + y_2 \end{pmatrix} \implies JF(\mathbf{y}) = \begin{pmatrix} \alpha + y_2^2 & 2 + 2y_1 y_2 \\ \alpha + 1 + 3\alpha y_1^2 & 1 \end{pmatrix},$$

hence at $\mathbf{y}_0 = (0,0)^T$: $JF(\mathbf{y}_0) = \begin{pmatrix} \alpha & 2 \\ \alpha + 1 & 1 \end{pmatrix}$.

Compute eigenvalues:

$$\det \begin{pmatrix} \alpha - \lambda & 2 \\ \alpha + 1 & 1 - \lambda \end{pmatrix} = 0 \iff \lambda^2 - (\alpha + 1)\lambda - (\alpha + 2) = 0$$

$$\implies \lambda_1 = -1, \; \lambda_2 = \alpha + 2.$$

Apply the Theorem: since $\Re(\lambda_1) = -1 < 0$, it is

- if $\alpha < -2$, $\Re(\lambda_{1/2}) < 0 \implies \mathbf{y}_0 = (0,0)^T$ is an asymptotically stable equilibrium;
- if $\alpha > -2$, $\Re(\lambda_2) > 0 \implies \mathbf{y}_0 = (0,0)^T$ is an unstable equilibrium.

Equilibrium of linear autonomous systems

Theorem (Stability criterion for linear autonomous systems)

Let $\mathbf{y}_0 \in \mathbb{R}^n$ be a stationary point for the linear system $\mathbf{y}'(t) = A \cdot \mathbf{y}(t)$ with $A \in \mathbb{R}^{n \times n}$, and let $\{\lambda_i\}_{i=1,\dots,n}$ be the eigenvalues of A. It holds:

- *if $\Re(\lambda_i) < 0$ for every $i \in \{1, \dots, n\} \implies \mathbf{y}_0$ is a point of **asymptotically stable** equilibrium*
- *if $\Re(\lambda_i) \leq 0$ for every $i \in \{1, \dots, n\}$ AND whenever $\Re(\lambda_i) = 0$ it is $g(\lambda_i) = a(\lambda_i) \implies \mathbf{y}_0$ is a point of **stable** equilibrium*
- *if there is at least one $i \in \{1, \dots, n\}$ s.t. $\Re(\lambda_i) > 0$ OR $\{\Re(\lambda_i) = 0$ with $g(\lambda_i) < a(\lambda_i)\} \implies \mathbf{y}_0$ is a point of **unstable** equilibrium.*

Remark on linear autonomous systems: without loss of generality we can reduce to homogeneous case (the type of stability is NOT affected when applying a translation!) and thus to $\mathbf{y}_0 = 0$ as stationary point.

Example 3

- Given the autonomous system $\mathbf{y}'(t) = A_1 \cdot \mathbf{y}(t)$ with $A_1 = \begin{pmatrix} -4 & 0 \\ 1 & 1 \end{pmatrix}$, the eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = 1$.

  Apply the stability criterion: since $\Re(\lambda_2) = 1 > 0$, $\mathbf{y}_0 = (0,0)^T$ is a point of unstable equilibrium.

- Considering instead $A_2 = \begin{pmatrix} -4 & -1 \\ 2 & -2 \end{pmatrix}$, we find $\det \begin{pmatrix} -4-\lambda & -1 \\ 2 & -2-\lambda \end{pmatrix} = \lambda^2 + 6\lambda + 10 = (\lambda + 3 + i)(\lambda + 3 - i)$, thus the eigenvalues are $\lambda_{1/2} = -3 \pm i$.

  Being $\Re(\lambda_1) < 0$ and $\Re(\lambda_2) < 0$, from the criterion the point $\mathbf{y}_0 = (0,0)^T$ is of asymptotically stable equilibrium for $\mathbf{y}'(t) = A_2 \cdot \mathbf{y}(t)$.

Exercises

- Exercise 1. Determine the solution of the following IVP applying the Laplace transform.

$$\begin{cases} y''(t) + y'(t) = f(t), \;\; t > 0; \\ y(0) = y'(0) = 0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(t) = 1$ in $1 \le t < 5$ and $f(t) = 0$ otherwise.

- Exercise 2. Determine the function $f : [0, \infty) \to \mathbb{R}$ such that its Laplace transform is

$$F(s) = \frac{8}{s^3(s+2)}, \quad \text{for (some) } s \in \mathbb{R}.$$

Hint: employ first partial fraction decomposition writing

$$\frac{8}{s^3(s+2)} = \frac{As^2 + Bs + C}{s^3} + \frac{D}{s+2},$$

for $A, B, C, D \in \mathbb{R}$ to be determined.

Exercises

- Exercise 3. Determine the stationary points and their equilibrium for the (non-linear) autonomous second order differential equation

$$\mathbf{y}''(t) + \mathbf{y}'(t) - \mathbf{y}(t)^2 + 1 = 0. \tag{5}$$

Hint: Rewrite (5) as system of two first-order ODEs.

- Exercise 4. For any of the following matrices A_i, analyze the stability of the corresponding linear homogeneous system represented by $\mathbf{y}'(t) = A_i \cdot \mathbf{y}(t)$.

(i) $A_1 = \begin{pmatrix} 0 & -1/2 \\ 18 & 0 \end{pmatrix}$;

(ii) $A_2^\alpha = \begin{pmatrix} \alpha & 0 & -1 \\ 1 & -5 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$;

(iii) $A_3^\alpha = \begin{pmatrix} i & 0 & 0 & 0 \\ 7 & -i & 0 & 0 \\ 0 & \alpha & i & 0 \\ -2 & 0 & 4 & -i \end{pmatrix}$, $\alpha \in \mathbb{R}$.

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EXERCISE 4

$Y'(t) = \underbrace{A}_{\in \mathcal{M}_{n\times n}(\mathbb{R})} \cdot Y(t) \to$ linear homogeneous system of n ODEs

Determine stability.

(i) $A = \begin{pmatrix} 0 & -1/2 \\ 18 & 0 \end{pmatrix}$ $Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to$ equilibrium point (since the system is homogeneous)

Compute eigenvalues of A: $\begin{vmatrix} -\lambda & -1/2 \\ 18 & -\lambda \end{vmatrix} = \lambda^2 + 9 = 0 \Leftrightarrow \lambda_{1/2} = \pm 3i$, $Re(\lambda_{1/2}) = 0 \Rightarrow$

$\Rightarrow$ need to check multiplicity, $a(\lambda_{1/2}) = 1 \geq g(\lambda_{1/2}) \geq 1 \Rightarrow$ geom. multiplicity is $1 =$ algebraic multiplicity!

$\Rightarrow$ (THM.) Y_0 stable equilibrium.

(ii) $A_\alpha = \begin{pmatrix} \alpha & 0 & -1 \\ 1 & -5 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$ $Y_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ equilibrium point

$\det \begin{pmatrix} \alpha-\lambda & 0 & -1 \\ 1 & -5-\lambda & 0 \\ 0 & 0 & \alpha-\lambda \end{pmatrix} = (\alpha-\lambda)(\alpha-\lambda)(-5-\lambda) = -(\lambda-\alpha)^{2}(\lambda+5) = 0$

$\nearrow \lambda_1 = \lambda_2 = \alpha$???

$\searrow \lambda_3 = -5 \sim Re(\lambda_3) = -5 < 0$ ✓

- If $\alpha > 0 \Rightarrow \lambda_{1/2} = \alpha$, $Re(\lambda_{1/2}) > 0 \overset{\text{THM.}}{\Rightarrow}$ unstable equilibrium
- If $\alpha < 0 \Rightarrow \lambda_{1/2} = \alpha < 0$, $Re(\lambda_{1/2}) < 0$, $Re(\lambda_3) < 0 \overset{\text{THM.}}{\Rightarrow}$ asymptotically stable equilibrium
- If $\alpha = 0 \Rightarrow Re(\lambda_{1/2}) = 0$, alg. multiplicity $a(0) = 2$, need to compute its geometric mult.!

$\det(A_0 - 0 I_3) v = 0 \sim \begin{pmatrix} 0 & 0 & -1 \\ 1 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \sim \begin{array}{l} -v_3 = 0 \\ v_1 - 5v_2 = 0 \\ \hookrightarrow v_1 = 5v_2 \end{array}$ $v = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$

$1 = g(0) < a(0) = 2 \overset{\text{THM.}}{\Rightarrow}$ unstable equilibrium.

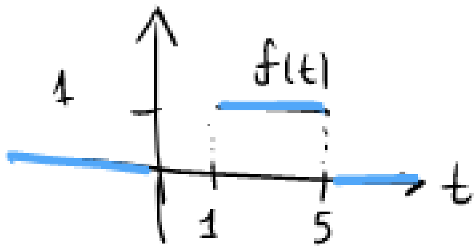
EXERCISE 1 $y = y(t)$, $f = f(t)$. Solve (IVP) applying the Laplace transform.

$$\begin{cases} y'' + y' = f, & t > 0 \\ y(0) = y'(0) = 0 \end{cases} \quad \text{(IVP)}$$

$f: \mathbb{R} \to \mathbb{R}$ s.t. $f(t) = \begin{cases} 1; & 1 \le t < 5 \\ 0; & \text{elsewhere} \end{cases}$

It is:

$$f(t) = h_1(t) - h_5(t) = \begin{cases} 0 - 0 = 0; & t < 1 \\ 1 - 0 = 1; & 1 \le t < 5 \\ 1 - 1 = 0; & t \ge 5 \end{cases}$$



where

$$h_a(t) = \begin{cases} 1; & t \ge a \\ 0; & t < a \end{cases} \quad \leftarrow \text{Heaviside function}$$

Apply Laplace transformation to (IVP): $\mathcal{L}[y'' + y'] = \mathcal{L}[f] = \mathcal{L}[h_1 - h_5]$

$\mathcal{L}[y''] + \mathcal{L}[y'] = \mathcal{L}[h_1] - \mathcal{L}[h_5]$ ↙ linearity of $\mathcal{L}$

$\mathcal{L}[y] =: Y(s)$

from the derivative formula

$\mathcal{L}[y'] = s \cdot Y(s) - y(0) = s \cdot Y(s)$

$\mathcal{L}[y''] = s^2 Y(s) - s \cdot y(0) - y'(0) = s^2 \cdot Y(s)$

$\mathcal{L}[h_1] = \frac{e^{-s}}{s}$ ← from table

$\mathcal{L}[h_5] = \frac{e^{-5s}}{s}$

$\Rightarrow s^2 Y(s) + s \cdot Y(s) = \frac{e^{-s} - e^{-5s}}{s}$, where $s^2 Y(s) + s \cdot Y(s) = s(s+1) Y(s)$

Solve for $Y(s)$: $Y(s) = \frac{e^{-s} - e^{-5s}}{s^2(s+1)}$, $s \neq 0, s \neq -1$

need to invert $Y(s) \multimap y(t)$

partial fraction decomposition

$$\frac{1}{s^2(s+1)} = \frac{As + B}{s^2} + \frac{C}{s+1} = \dots = \frac{1-s}{s^2} + \frac{1}{s+1} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

Hence:

$$Y(s) = (e^{-s} - e^{-5s})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) = -\frac{e^{-s}}{s} + \frac{e^{-5s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-5s}}{s^2} + \frac{e^{-s}}{s+1} - \frac{e^{-5s}}{s+1} \quad \leftarrow$$

Invert $\mathcal{L}$ on each term:

$\frac{e^{-s}}{s} \multimap h_1(t)$ | $\frac{e^{-5s}}{s} \multimap h_5(t)$ | $\frac{e^{-s}}{s^2} = e^{-s} \cdot \frac{1}{s^2}$ where $\frac{1}{s^2} \multimap t$ (table)

Apply the shifting rule:

$$e^{-sa} \mathcal{L}[g(t)] = \mathcal{L}[h_a(t) \cdot g(t-a)]$$

with $a = 1$ and $g(t) = t$;

$\frac{e^{-s}}{s^2} \multimap h_1(t) \cdot g(t-1) = h_1(t) \cdot (t-1)$

$\frac{e^{-5s}}{s^2} = e^{-5s} \cdot \frac{1}{s^2}$, $\frac{1}{s^2} \multimap t$; from with $a = 5$, $g(t) = t$

we find: $\frac{e^{-5s}}{s^2} \multimap h_5(t)(t-5)$

$\frac{e^{-s}}{s+1} = e^{-s} \cdot \frac{1}{s+1}$ with $\frac{1}{s+1} \multimap e^{-t}$ (table); from

with $a = 1$, $g(t) = e^{-t} \Rightarrow \frac{e^{-s}}{s+1} \multimap h_1(t)\, e^{-(t-1)} = h_1(t)\, e^{1-t}$

$\frac{e^{-5s}}{s+1} = e^{-5s} \cdot \frac{1}{s+1}$ with $\frac{1}{s+1} \multimap e^{-t}$; again by with $a = 5$, $g(t) = e^{-t} \Rightarrow \frac{e^{-5s}}{s+1} \multimap h_5(t)\, e^{5-t}$

$$Y(s) = \mathcal{L}\left[-h_1(t) + h_5(t) + h_1(t) \cdot (t-1) - h_5(t) \cdot (t-5) + h_1(t)\, e^{1-t} - h_5(t)\, e^{5-t}\right] =$$

$$= \mathcal{L}\left[h_1(t)\left[t - 1 - 1 + e^{1-t}\right] + h_5(t)\left[1 - t + 5 - e^{5-t}\right]\right] = \mathcal{L}\left[h_1(t)(t - 2 + e^{1-t}) + h_5(t)(6 - t - e^{5-t})\right] \Rightarrow$$

by existence and uniqueness thm

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = h_1(t)(t - 2 + e^{1-t}) + h_5(t)(6 - t - e^{5-t}), \ t \ge 0.$$

$\rightsquigarrow$ check that it satisfies indeed the (IVP)!